COMBINATORICA 3 (1) (1983) 119—121

ON PERIODICAL BEHAVIOUR IN SOCIETIES WITH SYMMETRIC INFLUENCES

Svatopluk POLJAK and Miroslav ŠŮRA

Received 3 March 1982

We propose a simple model of society with a symmetric function $w(u, v)$ measuring the influence of the opinion of member v on that of member u. The opinions are chosen from a finite set. At each step everyone accepts the majority opinion (with respect to w) of the other members. The behaviour of such a society is clearly periodic after some initial time. We prove that the length of the period is either one or two.

Let V be a finite set and $f_0\colon V\to\{0, 1, \dots, p\}$ be a mapping. Let us interpret the elements of V as members of a society and $f_0(u)$ as an initial opinion of each member $u\in V$. Let the real number $w(u, v)$ denote the influence of the member u on the member v for each pair $u, v\in V$ (u, v not necessarily distinct). We say that w is *symmetric* if $w(u, v)=w(v, u)$ for every $u, v\in V$.

The system (V, w, f_0) develops according to the following rules:

Every member accepts the majority opinion (with respect to the weights w) of other members (if there are two or more majority opinions, he accepts the highest one). This gives a new mapping $f_1; V\to\{0, 1, \dots, p\}$. The mapping f_1 determines a mapping f_2 by the same rule, and so on.

More precisely, for every $t=0, 1, \dots$ let us define the mapping $f_{t+1}\colon V\to\{0, 1, \dots, p)$ by

(1) $$f_{t+1}(u) = \max\Big\{i\colon\ \forall j \sum_{f_t(v)=i} w(u, v) \geqq \sum_{f_t(v)=j} w(u, v)\Big\}.$$

As there is only a finite number of distinct mappings from V to $\{0, 1, \dots, p\}$, and every f_t determines a unique mapping f_{t+1}, the system (V, w, f_0) will behave periodically beginning from some time t^*.

Let us define the period of (V, w, f_0) as

(2) $$\min\{t>0\colon\ f_{t^*+t}=f_{t^*}\ \text{ for some }\ t^*\}.$$

In this paper we prove the following result.

Theorem. *The period is* 1 *or* 2 *for any system* (V, w, f_0) *with symmetrical influences* w.

AMS subject classification (1980): 90 A 08

Proof. Let k be the period of the system (V, w, f_0) and t^* be such that $f_{t^*+k}=f_{t^*}$. Let us identify every vertex $u\in V$ with the vector $(u_1, u_2, \ldots, u_k)\in\{0, 1, \ldots, p\}^k$ whose components are $u_i=f_{t^*+i}(u)$, $i=1, \ldots, k$.

For every vertex $u\in V$ we can re-write (1) as a system $\Phi(u)$ of $k(p+1)$ inequalities

$$\tag{3} \begin{gathered} \sum_v \delta(v_{i-1}, u_i)w(v, u) \geqq \sum_v \delta(v_{i-1}, j)w(v, u) \quad \text{if} \quad j = 0, 1, \ldots, u_i, \\ \sum_v \delta(v_{i-1}, u_i)w(v, u) > \sum_v \delta(v_{i-1}, j)w(v, u) \quad \text{if} \quad j = u_i+1, \ldots, p \\ \text{for } i=1, 2, \ldots, k \text{ (the indices are } \bmod k), \end{gathered}$$

where the coefficient $\delta(x, y)$ is 1 if $x=y$, and 0 if $x\neq y$.

For every vertex $u\in V$ let us select from (3) a subsystem $\Phi'(u)$ of k inequalities:

$$\tag{4} \sum_v [\delta(v_{i-1}, u_i)-\delta(v_{i-1}, u_{i-2})]w(v, u) \begin{matrix} \geqq 0 & \text{if} & u_{i-2}\leqq u_i \\ > 0 & \text{if} & u_{i-2}> u_i \end{matrix} \qquad \text{for} \quad i = 1, 2, \ldots, k.$$

Let us assume that the period k is greater than 2. Then V must contain a vertex u not of the type $(i, i, \ldots, i)$ or $(i, j, i, j, \ldots, i, j)$. For such a vertex u it is clear that at least one inequality in (4) is sharp.

Summing all the inequalities of $\Phi'(u)$ over all $u\in V$ we get one sharp inequality

$$\tag{5} \sum_{(v,u)\in V\times V} A(v, u)w(v, u) > 0.$$

where

$$\tag{6} A(v, u) = \sum_{i=1}^{k} [\delta(v_{i-1}, u_i)-\delta(v_{i-1}, u_{i-2})].$$

For every pair $u, v\in V$ we have

$$A(v, u)+A(u, v) = \sum_{i=1}^{k} \delta(v_{i-1}, u_i)-\sum_{i=1}^{k} \delta(v_{i-1}, u_{i-2})+ \\ + \sum_{i=1}^{k} \delta(u_{i-1}, v_i)-\sum_{i=1}^{k} \delta(u_{i-1}, v_{i-2}) = 0.$$

As the influence function w is symmetric, we have $A(u, v)w(u, v)+ +A(v, u)w(v, u)=0$ for every pair $u, v\in V$, which contradicts inequality (5). Thus the assumption $k>2$ was false and this concludes the proof. ▮

Concluding remarks

1. In the model, a rule was accepted for tie-breaking: "the member accepts the highest of the majority opinions". This can be avoided (so that a tie never occurs) by changing the model slightly, e.g. by introducing

a) new vertices $v^0, v^1, \ldots, v^p$ with initial opinions $f(v^i)=i$ for $i=0, 1, \ldots, p$,
b) new loops with influences $w(v^i, v^i)=L$ (L sufficiently large) for every v^i,
c) influences $w(v^i, u)=\varepsilon \cdot i$ (ε sufficiently small) for every i and $u \in V$.

The new vertices $v^0, v^1, \ldots, v^p$ will keep their initial opinions. If in the original model a tie occurs, the modified model solves it in exactly the same way, using only simple majority rule.

2. If the influence function w is not symmetric, the period can be arbitrarily large.

3. Another model of social influences was introduced and studied by French [1] and Harary [2]. (For a survey, see the book [3]). The main differences of their model in comparison with ours are:
— "opinions" of the members $u \in V$ are *real* numbers $f_t(u)$,
— influences $w(u, v)$ between members are *nonnegative* real numbers,
— the next opinion of a member u is the *average* opinion of the others, i.e.

$$f_{t+1}(u) = \frac{\sum_v w(v, u) f_t(v)}{\sum_v w(v, u)}.$$

An example given in [3] considers an opinion to be a sum of money that should be spent on a particular project. This model is *quantitative,* involving distinct degrees of only one alternative. On the other hand, our model is *qualitative* since every member must accept exactly one of several alternatives.

References

[1] J. R. P. French, A Formal Theory of Social Power, *Psych. Review* **63** (1956), 181—194.
[2] F. Harary, A Criterion for Unanimity in French's Theory of Social Power, *in: Research,* Ann Arbor, Michigan, (1959), 168—182.
[3] F. S. Roberts, *Discrete Mathematical Models, with Application to Social, Biological, and Environmental Problems,* Prentice-Hall, Englewood Cliffs, N. J. 1976.

Svatopluk Poljak, Miroslav Sůra

Stavební fakulta ČVUT
Katedra ASŘ, Thákurova 7
16629 Praha, 6, Czechoslovakia